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# SCATTERING AND DIFFRACTION OF SH WAVES BY MULTIPLE PLANAR CRACKS IN AN ANISOTROPIC HALF-SPACE: A HYPERSINGULAR INTEGRAL FORMULATION

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**Abstract**—The scattering and diffraction of SH waves by arbitrarily-located multiple planar cracks in an anisotropic half-space is considered. The problem is formulated in terms of a system of Hadamard finite-part singular (hyper-singular) integral equations which can be solved numerically using collocation techniques.

## 1. INTRODUCTION

The interaction of harmonic SH waves with planar cracks in elastic media has been a research subject of considerable interest among many investigators. Using integral transforms, Loeber and Sih (1968) and Mal (1970) have solved the problem for a single planar crack in an isotropic medium. An extension of the work to orthotropic materials may be found in Ohyoshi (1973). For multiple interacting cracks, Jain and Kanwal (1972) and Itou (1980) have obtained approximate solutions to the problem for the special case involving a pair of coplanar cracks in an isotropic medium; more recently, Gross and Zhang (1988) have provided a method of solution for the more general case which concerns an arbitrary number of arbitrarily-oriented planar cracks in an isotropic medium, and, using a dislocation model, So and Huang (1988) have developed a numerical procedure for the case of two arbitrarily-oriented planar cracks in an isotropic elastic space.

The work in the references cited above is confined to (unbounded) elastic full space. In the present paper, the scattering and diffraction of SH waves by arbitrarily-located multiple planar cracks in a general anisotropic half-space is considered. Using an approach similar to that given by Gross and Zhang (1988), we formulate the problem under consideration in terms of a system of Hadamard finite-part singular (hypersingular) integral equations which are readily amenable to numerical treatment. Numerical results are obtained for some specific cases of the problem.

## 2. STATEMENT OF THE PROBLEM

Referring to a Cartesian coordinate system $0x_1x_2x_3$, consider an anisotropic half-space which occupies the region $x_2 > 0$. The interior of the half-space contains $N$ arbitrarily-located planar cracks with geometries that do not vary with the coordinate $x_3$. We assume that the cracks do not intersect one another or the boundary $x_2 = 0$. Denote the $i$th crack by $\Gamma^{(i)}$. On the plane $x_3 = 0$, the tips of the crack $\Gamma^{(i)}$ are given by $(\alpha^{(i)}, \beta^{(i)})$ and $(\gamma^{(i)}, \delta^{(i)})$ (Fig. 1).

A horizontally polarized SH (plane) wave with displacement in the $0x_3$ direction (of unit amplitude)

$$u_{3I} = \exp[i\omega(t + c_1^{-1}x_1 + c_2^{-1}x_2)], \tag{1}$$

propagates towards the boundary $x_2 = 0$ of the anisotropic half-space. Note that

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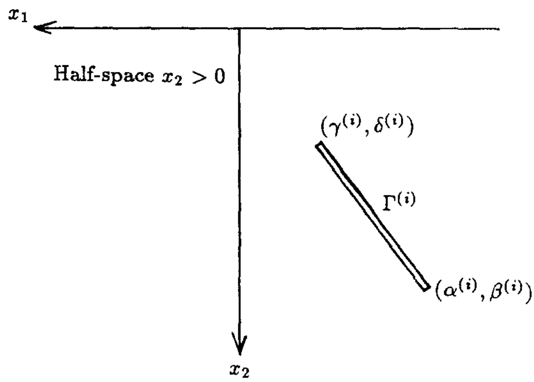


Fig. 1. A typical planar crack in an elastic half-space.

$\mathrm{i} = \sqrt{-1}$, $\omega$ is the circular frequency of the incident wave, $u_3$ denotes the displacement in the $0x_3$ direction, $t$ is the time coordinate, and $c_1$ and $c_2$ are constants.

Our problem is to determine the displacement and the stress fields associated with the scattered and diffracted waves, given that the cracks and the boundary $x_2 = 0$ remain free of traction at all time.

## 3. DISPLACEMENT FIELD IN AN UNCRACKED HALF-SPACE

For the problem described in Section 2 above, the only non-zero displacement is in the $0x_3$ direction and given by $u_3 = u_3(x_1, x_2, t)$. It follows that the equation of motion is given by

$$\lambda_{ij} \frac{\partial^2 u_3}{\partial x_i \partial x_j} = \rho \frac{\partial^2 u_3}{\partial t^2}, \tag{2}$$

where $\lambda_{ij}$ are the elastic moduli of the anisotropic material in the half-space, with $\lambda_{ij} = \lambda_{ji}$ and $\lambda_{12}^2 - \lambda_{11}\lambda_{22} < 0$, and $\rho$ is the density of the material. Note that, unless otherwise stated, the usual convention of summing over a repeated index is adopted only for Latin subscripts which are assumed to take the values of 1 and 2.

Now, the displacement field in (1) must satisfy the governing equation (2). Hence,

$$c_1^{-2}\lambda_{11} + 2(c_1 c_2)^{-1}\lambda_{12} + c_2^{-2}\lambda_{22} = \rho. \tag{3}$$

If the incident wave with the displacement field (1) has an angle of incident $\theta_1$ as depicted in Fig. 2 then $c_1 = v/\sin(\theta_1)$ and $c_2 = v/\cos(\theta_1)$, where, from (3), the constant $v$ is found to be

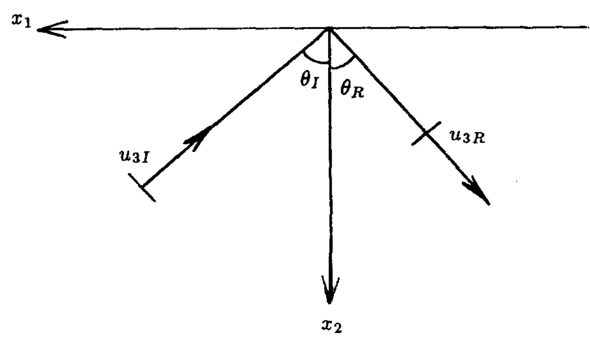


Fig. 2. Angles of incidence and reflection.

$$v^2 = \rho^{-1}\{\lambda_{11}\sin^2(\theta_I)+2\lambda_{12}\sin(\theta_I)\cos(\theta_I)+\lambda_{22}\cos^2(\theta_I)\}. \tag{4}$$

Note that $v$ is interpreted as the wave speed of the incident wave.

To obtain the displacement field in the anisotropic half-space for the special case where there is no crack present, we write:

$$u_3 = u_{3I}+u_{3R}, \tag{5}$$

where $u_{3I}$ is as given in (1) and $u_{3R}$ represents the displacement associated with the reflected wave and takes the form:

$$u_{3R} = \exp[i\omega(t+\{c_1'\}^{-1}x_1-\{c_2'\}^{-1}x_2)], \tag{6}$$

where $c_1'$ and $c_2'$ are constants.

Using the linear stress–strain relation

$$\sigma_{k3} = \lambda_{kj}\frac{\partial u_3}{\partial x_j}, \tag{7}$$

together with (1), (5) and (6), we obtain

$$\sigma_{k3} = i\omega(c_1^{-1}\lambda_{k1}+c_2^{-1}\lambda_{k2})\exp[i\omega(t+c_1^{-1}x_1+c_2^{-1}x_2)]$$
$$+i\omega(\{c_1'\}^{-1}\lambda_{k1}-\{c_2'\}^{-1}\lambda_{k2})\exp[i\omega(t+\{c_1'\}^{-1}x_1-\{c_2'\}^{-1}x_2)]. \tag{8}$$

From (8), we find that the condition that the boundary $x_2 = 0$ is traction-free holds if we choose $c_1'$ and $c_2'$ to be given by

$$c_1' = c_1 \quad \text{and} \quad \{c_2'\}^{-1} = c_2^{-1}+2(c_1\lambda_{22})^{-1}\lambda_{12}. \tag{9}$$

It is an easy matter to verify by direct substitution that, with $c_1'$ and $c_2'$ as given by (9), the displacement $u_{3R}$ in (6) [and hence $u_3$ in (5)] satisfies (2) exactly.

Letting $c_1' = v'/\sin(\theta_R)$ and $c_2' = v'/\cos(\theta_R)$, where $\theta_R$ is the angle of reflection as depicted in Fig. 2 and $v'$ is the wave speed of the reflected wave, and using (9), we obtain

$$\tan(\theta_R) = (1+2\lambda_{22}^{-1}\lambda_{12}\tan(\theta_I))^{-1}\tan(\theta_I). \tag{10}$$

Note that for media where $\lambda_{12} = 0$ (for example, isotropic media) $\theta_I = \theta_R$.

## 4. SOLUTION OF THE CRACK PROBLEM

In view of the forms of the incident and reflected waves in (1) and (6), respectively, a solution to the crack problem in Section 2 may be sought in the form:

$$u_3(x_1,x_2,t) = u(x_1,x_2)\exp(i\omega t). \tag{11}$$

Substituting (11) into (2), we find that $u(x_1, x_2)$ in (11) is required to satisfy

$$\lambda_{ij}\frac{\partial^2 u}{\partial x_i\,\partial x_j}+\rho\omega^2 u = 0. \tag{12}$$

Let $\Omega$ be a two-dimensional finite region (in the $0x_1x_2$ plane) with boundary $\partial\Omega$. If eqn (12) is valid in $\Omega$ then an integral solution to (12) is given by (Clements and Larsson, 1991)

$$Cu(\xi_1,\xi_2) = \int_{\partial\Omega} \lambda_{ij}\left[u(x_1,x_2)\frac{\partial V}{\partial x_j} - V\frac{\partial}{\partial x_j}u(x_1,x_2)\right]n_i(x_1,x_2)\,\mathrm{d}S, \tag{13}$$

where $C = 1$ if $(\xi_1, \xi_2) \in \Omega$ and $0 < C < 1$ if $(\xi_1, \xi_2) \in \partial\Omega$; $n_i = n_i(x_1, x_2)$ are the components of the unit normal outward vector to $\partial\Omega$ at $(x_1, x_2)$; and $V = V(x_1, x_2, \xi_1, \xi_2)$ is, for our purpose here, chosen to be

$$V = \frac{\mathrm{i}K}{4}[H_0^{(2)}(\omega^* R) + H_0^{(2)}(\omega^* R^*)], \tag{14}$$

where $H_0^{(2)}(x)$ is the zeroth order Hankel function of the second kind, $(\omega^*)^2 = \omega^2\rho K$ and

$$K = 2[\lambda_{11} + \lambda_{12}(\tau+\bar{\tau}) + \tau\bar{\tau}\lambda_{22}]^{-1},$$

$$R = \sqrt{(x_1-\xi_1+\tfrac{1}{2}[\tau+\bar{\tau}][x_2-\xi_2])^2 + \left(\frac{\mathrm{i}}{2}[\bar{\tau}-\tau][x_2-\xi_2]\right)^2},$$

$$R^* = \sqrt{(x_1-\xi_1+\tfrac{1}{2}[\tau+\bar{\tau}][x_2-\xi_2])^2 + \left(\frac{\mathrm{i}}{2}[\bar{\tau}-\tau][x_2+\xi_2]\right)^2}, \tag{15}$$

with the constant $\tau$ being a complex number having positive imaginary part and satisfying

$$\lambda_{11} + 2\lambda_{12}\tau + \lambda_{22}\tau^2 = 0. \tag{16}$$

Note that the inequality $\lambda_{12}^2 - \lambda_{11}\lambda_{22} < 0$ guarantees that $\tau$ in (16) has a non-zero imaginary part. The bar denotes the complex conjugate of a complex number.

It is an easy matter to verify that the choice of $V = V(x_1, x_2, \xi_1, \xi_2)$ in (14) satisfies the condition

$$\lambda_{2j}\frac{\partial V}{\partial x_j} = 0 \quad \text{on the plane } x_2 = 0. \tag{17}$$

Now, for the solution of the crack problem described in Section 2, let

$$u = u^{(1)} + u^{(2)}, \tag{18}$$

where $u^{(1)}$ denotes the displacement field in the half-space for the special case where there is no crack present and $u^{(2)}$ denotes the displacement field imduced by the presence of the cracks.

From (1), (5) and (6), the displacement $u^{(1)}$ is given by

$$u^{(1)} = \exp[\mathrm{i}\omega(c_1^{-1}x_1 + c_2^{-1}x_2)] + \exp[\mathrm{i}\omega(\{c_1'\}^{-1}x_1 - \{c_2'\}^{-1}x_2)], \tag{19}$$

where $c_1$, $c_2$, $c_1'$ and $c_2'$ are as determined in Section 3.

The displacement $u^{(2)}$ is required to satisfy (12) subject to the conditions:

$$p^{(2)}(x_1, 0) = 0 \quad \text{for } -\infty < x_1 < \infty, \tag{20}$$

and

$$p^{(2)}(x_1,x_2) \to -p^{(1)}(x_1,x_2) \quad \text{as } (x_1,x_2) \to (\xi_1,\xi_2) \in \Gamma^{(k)} \quad (k = 1,2,\ldots,N), \tag{21}$$

where $p^{(m)} = \lambda_{ij}n_i\partial u^{(m)}/\partial x_j$ $(m = 1, 2)$ are the tractions which correspond to $u^{(m)}$. In addition, we shall assume that $u^{(2)}$ decays faster than $0(|(x_1, x_2)|^{-r})$, $r > 0$ is a real number, as $|(x_1, x_2)| \to \infty$. This assumption is used in the derivation of an integral expression for

$u^{(2)}(\xi_1, \xi_2)$ as in (22) below. In fact, we observe that the right-hand side of (22) behaves as $0(|(\xi_1, \xi_2)|^{-3/2})$ as $|(\xi_1, \xi_2)| \to \infty$. Thus, $r = 3/2$.

If we use (13), (14), (17) and (20), we may write

$$Cu^{(2)}(\xi_1, \xi_2) = \sum_{p=1}^{N} \int_{\Gamma_+^{(p)}} \lambda_{ij} \Delta u(x_1, x_2) \frac{\partial V}{\partial x_j} n_i(x_1, x_2)\,\mathrm{d}S, \tag{22}$$

where $\Delta u(x_1, x_2) = [u^{(2)}(x_1, x_2)]^+ - [u^{(2)}(x_1, x_2)]^-$ and $[f(x_1, x_2)]^+$ denotes the value of $f(x_1, x_2)$ on one face $\Gamma_+^{(i)}$ of the $i$th crack while $[f(x_1, x_2)]^-$ denotes the value of $f(x_1, x_2)$ on the other face $\Gamma_-^{(i)}$. We define $\Gamma_+^{(i)}$ to be the path from $(\alpha^{(i)}, \beta^{(i)})$ to $(\gamma^{(i)}, \delta^{(i)})$. Note that, for convenience, we may view $\Gamma^{(i)} = \Gamma_+^{(i)} \cup \Gamma_-^{(i)}$ as a closed path, assigned a clockwise direction, which encloses an elliptical region having an area which tends to zero.

From (22), for $(\xi_1, \xi_2)$ lying in the interior of the half-space, we may write

$$\lambda_{km} \frac{\partial}{\partial \xi_m} u^{(2)}(\xi_1, \xi_2) = \sum_{p=1}^{N} \int_{\Gamma_+^{(p)}} \lambda_{km} \lambda_{ij} \Delta u(x_1, x_2) G_{jm} n_i(x_1, x_2)\,\mathrm{d}S, \tag{23}$$

where $G_{jm} = G_{jm}(x_1, x_2, \xi_1, \xi_2) = \partial^2 V/\partial x_j\, \partial \xi_m$.

Now, if we let the point $(\xi_1, \xi_2)$ in (23) approach the crack face $\Gamma_+^{(n)}$ and apply (21) then we obtain the system of integral equations:

$$\begin{aligned}
&\mathscr{H}\int_{-1}^{1} \lambda_{km}\lambda_{ij} B_{jm}^{(n)} N_i^{(n)} N_k^{(n)} L^{(n)} (t-u)^{-2} \Delta U^{(n)}(t)\,\mathrm{d}t \\
&+ \int_{-1}^{1} \lambda_{km}\lambda_{ij} I_{jm}^{(n)}(t,u) N_i^{(n)} N_k^{(n)} L^{(n)} \Delta U^{(n)}(t)\,\mathrm{d}t \\
&+ \sum_{p=1, p\neq n}^{N} \int_{-1}^{1} \lambda_{km}\lambda_{ij} G_{jm}^{(pn)}(t,u) N_i^{(p)} N_k^{(n)} L^{(p)} \Delta U^{(p)}(t)\,\mathrm{d}t = \\
&-2P^{(n)}(u) \quad \text{for } -1 < u < 1 \quad (n = 1, 2, \ldots, N),
\end{aligned} \tag{24}$$

where $\mathscr{H}\int$ denotes that the integral (over the appropriate interval) must be interpreted in the Hadamard finite-part sense and

$$\Delta U^{(p)}(t) = \Delta u(X_1^{(p)}(t), X_2^{(p)}(t)),$$

$$G_{jm}^{(pn)}(t,u) = G_{jm}(X_1^{(p)}(t), X_2^{(p)}(t), X_1^{(n)}(u), X_2^{(n)}(u)),$$

$$H_{jm}(x_1, x_2, \xi_1, \xi_2) = \frac{\mathrm{i}K}{4} \frac{\partial^2}{\partial x_j\, \partial \xi_m} [H_0^{(2)}(\omega^* R^*)],$$

$$\begin{aligned}
I_{jm}^{(n)}(t,u) &= H_{jm}(X_1^{(n)}(t), X_2^{(n)}(t), X_1^{(n)}(u), X_2^{(n)}(u)) \\
&+ \frac{\mathrm{i}K}{4}(\omega^*)^2 \left[ C_{jm}^{(n)} + D_{jm} \ln \left\{ \frac{1}{2}\omega^* |t-u| \sqrt{\{\phi^{(n)}\}^2 + \left( \frac{\mathrm{i}}{4}[\bar{\tau}-\tau]\{\delta^{(n)} - \beta^{(n)}\} \right)^2} \right\} \right],
\end{aligned}$$

$$P^{(n)}(t) = \lambda_{km} N_k^{(n)} \left[ \frac{\partial u^{(1)}}{\partial x_m} \right]_{(X_1^{(n)}(t), X_2^{(n)}(t))},$$

$$B_{11}^{(n)} = \varphi^{(n)} \left[ \{\phi^{(n)}\}^2 - \left( \frac{\mathrm{i}}{4}[\bar{\tau}-\tau]\{\delta^{(n)} - \beta^{(n)}\} \right)^2 \right],$$

$$\begin{aligned}
B_{12}^{(n)} = B_{21}^{(n)} = \varphi^{(n)} \Bigg[ &-\frac{1}{2}[\tau+\bar{\tau}] \left\{ \{\phi^{(n)}\}^2 + \left( \frac{\mathrm{i}}{4}[\bar{\tau}-\tau]\{\delta^{(n)} - \beta^{(n)}\} \right)^2 \right\} \\
&+ \phi^{(n)} \left( [\tau+\bar{\tau}]\phi^{(n)} + \frac{1}{4}(\mathrm{i}[\tau-\bar{\tau}])^2 \{\delta^{(n)} - \beta^{(n)}\} \right) \Bigg],
\end{aligned}$$

$$B_{22}^{(n)} = \varphi^{(n)}\left[-\bar{\tau}\tau\left(\{\phi^{(n)}\}^2+\left(\frac{i}{4}[\bar{\tau}-\tau]\{\delta^{(n)}-\beta^{(n)}\}\right)^2\right)\right.$$

$$+\left([\tau+\bar{\tau}]\phi^{(n)}+\frac{1}{4}(i[\bar{\tau}-\tau])^2\{\delta^{(n)}-\beta^{(n)}\}\right)$$

$$\left.\times\left(\frac{1}{4}[\tau+\bar{\tau}]\{\gamma^{(n)}-\alpha^{(n)}\}+\frac{1}{2}\tau\bar{\tau}\{\delta^{(n)}-\beta^{(n)}\}\right)\right],$$

$$C_{11}^{(n)} = \frac{1}{2}+\frac{i}{\pi}\left(\frac{1}{2}-\gamma^*\right)-\frac{i}{\pi}\{\phi^{(n)}\}^2\sqrt{\frac{2\pi\varphi^{(n)}}{K}},$$

$$C_{12}^{(n)} = C_{21}^{(n)} = \frac{1}{2}[\bar{\tau}+\tau]\left(\frac{1}{2}+\frac{i}{\pi}\left[\frac{1}{2}-\gamma^*\right]\right)$$

$$-\frac{i}{\pi}\phi^{(n)}\sqrt{\frac{2\pi\varphi^{(n)}}{K}}\left(\frac{1}{2}[\bar{\tau}+\tau]\phi^{(n)}+\frac{1}{8}(i[\bar{\tau}-\tau])^2\{\delta^{(n)}-\beta^{(n)}\}\right),$$

$$C_{22}^{(n)} = \tau\bar{\tau}\left(\frac{1}{2}+\frac{i}{\pi}\left[\frac{1}{2}-\gamma^*\right]\right)$$

$$-\frac{i}{\pi}\sqrt{\frac{2\pi\varphi^{(n)}}{K}}\left(\frac{1}{2}[\tau+\bar{\tau}]\phi^{(n)}+\frac{1}{8}(i[\bar{\tau}-\tau])^2\{\delta^{(n)}-\beta^{(n)}\}\right)^2,$$

$$D_{11} = -\frac{i}{\pi},\quad D_{12} = D_{21} = \frac{1}{2}[\tau+\bar{\tau}]D_{11},\quad D_{22} = \tau\bar{\tau}D_{11}, \tag{25}$$

with $2X_1^{(n)}(t) = (\gamma^{(n)}+\alpha^{(n)})+(\gamma^{(n)}-\alpha^{(n)})t$, $2X_2^{(n)}(t) = (\delta^{(n)}+\beta^{(n)})+(\delta^{(n)}-\beta^{(n)})t$, $N_1^{(n)} = (\delta^{(n)}-\beta^{(n)})/L^{(n)}$, $N_2^{(n)} = (\alpha^{(n)}-\gamma^{(n)})/L^{(n)}$, $L^{(n)}$ is the length of the crack $\Gamma^{(n)}$,

$$\phi^{(n)} = \frac{1}{2}\{\gamma^{(n)}-\alpha^{(n)}\}+\frac{1}{4}[\tau+\bar{\tau}]\{\delta^{(n)}-\beta^{(n)}\},$$

$$\varphi^{(n)} = \frac{K}{2\pi}\left[\{\phi^{(n)}\}^2+\left(\frac{i}{4}[\bar{\tau}-\tau]\{\delta^{(n)}-\beta^{(n)}\}\right)^2\right]^{-2}$$

and $\gamma^* = 0.5772156649\ldots$ (Euler's constant). To explain why the integral with the $1/(t-u)^2$ singularity can be interpreted in the Hadamard finite-part sense, we only need to observe that the integral on the right-hand side of (22) is Cauchy principal as $(\xi_1, \xi_2)$ approaches the crack faces, and then apply eqn (11) in the paper by Kaya and Erdogan (1987).

Equation (24) constitutes a system of Hadamard finite-part singular (hyper-singular) integral equations from which we may solve for $\Delta u(x_1, x_2)$ for $(x_1, x_2)\in\Gamma_+^{(p)}$ (for $p = 1, 2, \ldots, N$). Consequently, once we solve these integral equations, we may compute the values of $u^{(2)}$ and $p^{(2)}$ using (22) and (23), respectively.

We shall now describe a procedure given in Kaya and Erdogan (1987) which may be used to obtain a numerical solution for (24).

Assume that (24) admits a solution in the approximate form

$$\Delta U^{(q)}(t) \approx \sqrt{1-t^2}\sum_{m=1}^{M} a_m^{(q)} U_{m-1}(t)\quad \text{for } -1 < t < 1\quad (q = 1, 2, \ldots, N), \tag{26}$$

where $U_m(x)$ is the $m$th order Chebyshev polynomial of the second kind and $a_m^{(q)}$ $(q = 1, 2, \ldots, N; m = 1, 2, \ldots, M)$ are constant coefficients (yet to be determined).

Substituting (26) into (24), we obtain

$$\sum_{q=1}^{N}\sum_{m=1}^{M} a_m^{(q)}\{-\delta_{qn}\lambda_{kp}\lambda_{ij}B_{jp}^{(n)}N_i^{(n)}N_k^{(n)}L^{(n)}m\pi U_{m-1}(s)+J_m^{(qn)}(s)\} = -2P^{(n)}(s) \quad \text{for } -1<s<1 \quad (n=1,2,\ldots,N), \quad (27)$$

where $\delta_{qn}$ denotes the Kronecker delta and

$$J_m^{(qn)} = \lambda_{kp}\lambda_{ij}N_k^{(n)}\int_{-1}^{1}[\delta_{qn}N_i^{(n)}L^{(n)}I_{jp}^{(n)}(t,s) + (1-\delta_{qn})N_i^{(q)}L^{(q)}G_{jp}^{(qn)}(t,s)]U_{m-1}(t)\sqrt{1-t^2}\,\mathrm{d}t. \quad (28)$$

Now if we select $s$ in (27) to be given in turn by

$$s = s_j = \cos(\{2j-1\}\pi/(2M)) \quad \text{for } j=1,2,\ldots,M, \quad (29)$$

then we find that (27) gives rise to a system of $MN$ linear algebraic equations in ($MN$ unknowns) $a_m^{(n)}$. The constant coefficients $a_m^{(n)}$ can then be determined by solving these linear algebraic equations.

## 5. NUMERICAL RESULTS

For the purpose of obtaining some numerical results from (26)–(29), we shall examine a few specific cases of the problem for a transversely-isotropic material with $\lambda_{12} = \lambda_{21} = 0$. In all the computations carried out below, we use the constants for the crystal of titanium, i.e. $\lambda_{11} = 0.47$, $\lambda_{22} = 0.35$ and $\rho = 4.51$ (the units for $\lambda_{ij}$ and $\rho$ are respectively in g $\text{cm}^{-1}$ $\mu\text{s}^{-1}$ $\text{cm}^{-3}$).

### 5.1. *A single planar crack in a transversely-isotropic half-space*

Take the crack tips of the single planar crack to be given by $(\alpha^{(1)}, \beta^{(1)}) = (0, d-\frac{1}{2}L)$ and $(\gamma^{(1)}, \delta^{(1)}) = (0, d+\frac{1}{2}L)$, where $L>0$ and $d>L/2$ are real numbers.

We compute the crack displacement difference $\Delta U^{(1)}(t)/(\Omega L)(-1<t<1)$ for $d/L = 1.000$, $\theta_1 = \pi/4$ and a few selected values of the non-dimensionalized wave number $\Omega$ (defined by $\Omega = \omega L/v$). The magnitude of the non-dimensionalized crack opening displacement $|\Delta U^{(1)}(t)/(\Omega L)|$ is plotted against $t(-1<t<1)$ for $\Omega = 1.000$, 5.000 and 10.000 in Figs 3–5, respectively. We vary the number of terms $M$ in (26) in order to investigate the convergence of the solution. From Figs 3–5, it is clear that, as $\Omega$ increases, more terms are required in the solution (26) to ensure satisfactory accuracy.

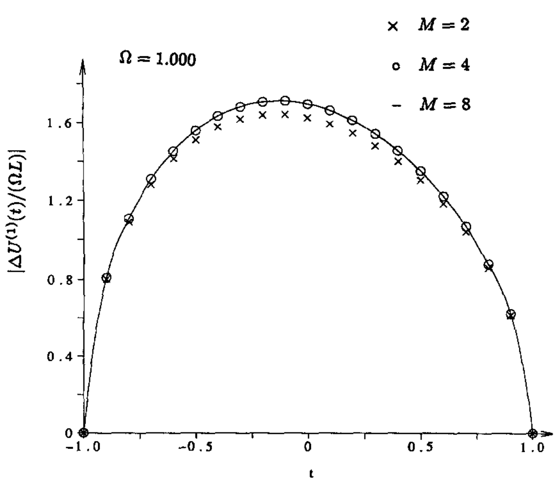


Fig. 3. Plots of $|\Delta U^{(1)}(t)/(\Omega L)|$ against $t$ for different $M$ ($\Omega = 1.000$).

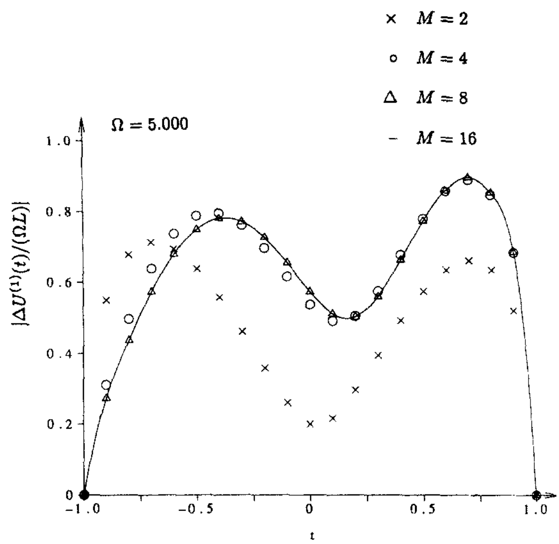


Fig. 4. Plots of $|\Delta U^{(1)}(t)/(\Omega L)|$ against $t$ for different $M$ ($\Omega = 5.000$).

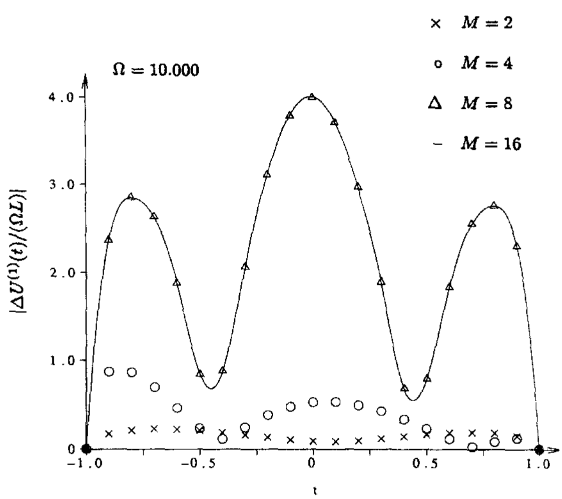


Fig. 5. Plots of $|\Delta U^{(1)}(t)/(\Omega L)|$ against $t$ for different $M$ ($\Omega = 10.000$).

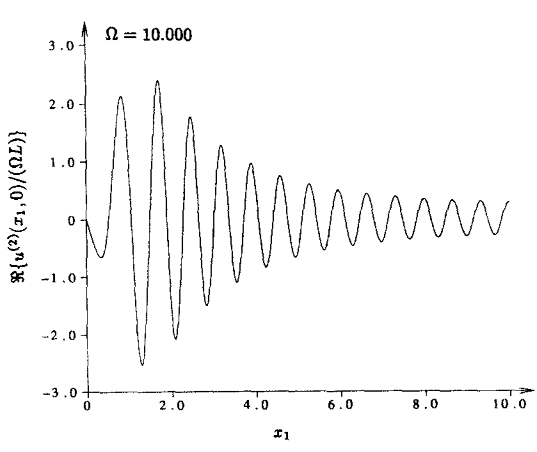


Fig. 6. Plot of the real part of the free surface displacement ($\Omega = 10.000$).

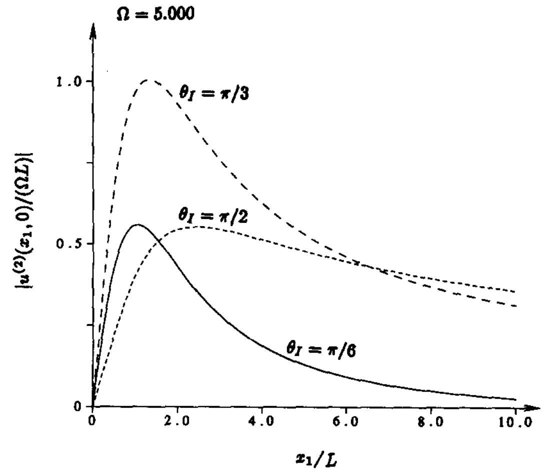


Fig. 7. Plot of the magnitude of the free surface displacement ($\Omega = 5.000$).

Selecting $d/L = 1.000$, we now compute the displacement $u^{(2)}(x_1, x_2)$ on the free surface $x_2 = 0$ by using (22). In Fig. 6, we plot $\Re\{u^{(2)}(x_1, 0)/(\Omega L)\}$ against $x_1 (0 \leqslant x_1/L \leqslant 10)$ for $\Omega = 10.000$ and $\theta_I = \pi/4$. The corresponding imaginary part of the displacement exhibits a similar variational pattern. For $\Omega = 5.000$, Fig. 7 shows the graphs of $|u^{(2)}(x_1, 0)/(\Omega L)|$ against $x_1 (0 \leqslant x_1/L \leqslant 10)$ for $\theta_I = \pi/6$, $\pi/3$ and $\pi/2$. In all the cases studied, we observe that the value of $u^{(2)}(x_1, 0)$ approaches 0 as $|x_1|$ becomes larger.

### 5.2. *Two planar cracks in a transversely-isotropic half-space*

Consider the case of a pair of parallel planar cracks with crack tips given by $(\alpha^{(1)}, \beta^{(1)}) = (-\frac{1}{2}L, d_1)$, $(\gamma^{(1)}, \delta^{(1)}) = (\frac{1}{2}L, d_1)$, $(\alpha^{(2)}, \beta^{(2)}) = (-\frac{1}{2}L, d_2)$ and $(\gamma^{(2)}, \delta^{(2)}) = (\frac{1}{2}L, d_2)$, where $d_1 > 0$ and $d_2 > d_1$ are given constants. We compute the magnitude of the non-dimensionalized crack displacement difference $|\Delta U^{(n)}(t)/(\Omega^2 L)| (n = 1, 2)$. Choosing $\theta_I = \pi/4$, $d_1/L = 0.500$ and $\Omega = 10.000$, we plot the results obtained for $d_2/L = 1.000$, $2.000$ and $3.000$ in Figs 8 and 9 ($|\Delta U^{(1)}(t)/(\Omega^2 L)|$ and $|\Delta U^{(2)}(t)/(\Omega^2 L)|$, respectively). (Note that for this particular case $P^{(n)}(s)$ behaves as $0(\Omega^2)$ as $\Omega \to 0^+$. Thus, we choose to divide $\Delta U^{(n)}(t)$ by $\Omega^2$ instead of $\Omega$.)

For the crack $\Gamma^{(1)}$, we define the crack-tip stress intensity factors:

$$k^- = \lim_{\xi \to -(L/2)^-} \lambda_{22}\sqrt{2(-\xi - \{L/2\})}\frac{\partial}{\partial \xi_2} u^{(2)}(\xi_1, \xi_2)|_{(\xi, d_1)},$$

$$k^+ = \lim_{\xi \to (L/2)^+} \lambda_{22}\sqrt{2(\xi - \{L/2\})}\frac{\partial}{\partial \xi_2} u^{(2)}(\xi_1, \xi_2)|_{(\xi, d_1)}. \tag{30}$$

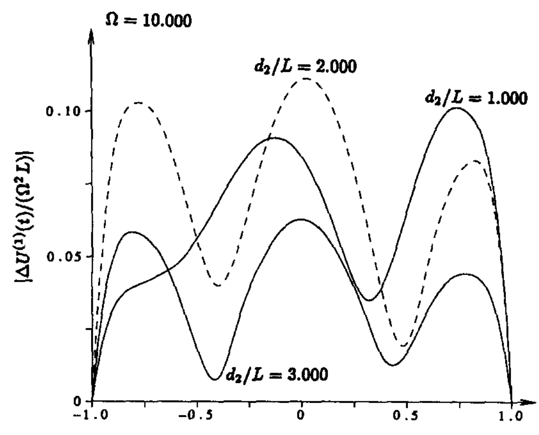


Fig. 8. Plots of $|\Delta U^{(1)}(t)/(\Omega^2 L)|$ against $t$ for various values of $d_2/L$ ($\Omega = 10.000$).

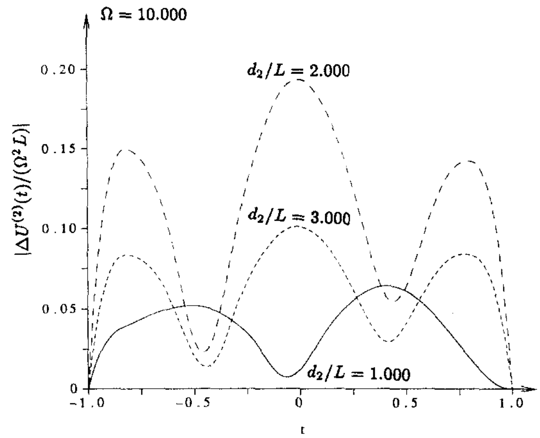


Fig. 9. Plots of $|\Delta U^{(2)}(t)/(\Omega^2 L)|$ against $t$ for various values of $d_2/L$ ($\Omega = 10.000$).

From the analysis in Section 4, we obtain

$$k^{\pm} \approx -\frac{\sqrt{2}}{8}\lambda_{22}^2 B_{22}^{(1)} L^{3/2}\pi \sum_{m=1}^{M} a_m^{(1)} U_{m-1}(\pm 1). \tag{31}$$

Taking $d_1/L = 0.500$, $d_2/L = 1.000$ and $\theta_1 = \pi/4$, we compute the non-dimensionalized stress intensity factors $k^{\pm}/(\Omega^2\lambda_{22}L^{1/2})$ and plot their magnitudes against $\Omega$ ($0 < \Omega \leqslant 10$) in Fig. 10. Note, that, as $\Omega \to 0^+$, $k^+/\Omega^2 \to k^-/\Omega^2$, as expected.

We now rotate the crack $\Gamma^{(2)}$ about its midpoint by an angle of $\eta$ in the anticlockwise direction, i.e. the tips of the crack $\Gamma^{(2)}$ are given by $(\alpha^{(2)}, \beta^{(2)}) = (-\frac{1}{2}L\cos(\eta),\ d_2 - \frac{1}{2}L\sin(\eta))$ and $(\gamma^{(2)}, \delta^{(2)}) = (\frac{1}{2}L\cos(\eta),\ d_2 + \frac{1}{2}L\sin(\eta))$, with $d_2 - d_1 > (L/2)\sin(\eta)$. For $d_1/L = 0.5000$, $d_2/L = 1.500$, $\theta_1 = 0$ and $\Omega = 1.000$, we compute the stress intensity factors $k^{\pm}$ for the crack $\Gamma^{(1)}$ [as defined in (31)] for varying $\eta$. We plot $|k^{\pm}/(\Omega^2\lambda_{22}L^{1/2})|$ against $\eta$ ($0 \leqslant \eta \leqslant \pi$) in Fig. 11. Note that $|k^+| \leqslant |k^-|$ for $0 \leqslant \eta \leqslant \pi/2$ and $|k^+| \geqslant |k^-|$ for $\pi/2 \leqslant \eta \leqslant \pi$.

### 5.3. *Three coplanar cracks in a transversely-isotropic half-space*

Three coplanar cracks which are parallel to the $0x_1$ axis lie in the interior of the half-space. Specifically, we take $(\alpha^{(1)}, \beta^{(1)}) = (-\frac{1}{2}L, d)$, $(\gamma^{(1)}, \delta^{(1)}) = (\frac{1}{2}L, d)$, $(\alpha^{(2)}, \beta^{(2)}) = (s + \frac{1}{2}L, d)$, $(\gamma^{(2)}, \delta^{(2)}) = (s + \frac{3}{2}L, d)$, $(\alpha^{(3)}, \beta^{(3)}) = (-s - \frac{1}{2}L, d)$ and $(\gamma^{(3)}, \delta^{(3)}) = (-s - \frac{3}{2}L, d)$, where $d$ and $s$ are positive constants. A SH wave impinges normally on the faces of the cracks. Choosing $d/L = 0.500$, we compute the stress intensity factors $k^{\pm}/(\Omega^2\lambda_{22}L^{1/2})$ for the crack $\Gamma^{(1)}$. (For this particular case, $k^+ = k^-$.) We plot $|k^+/(\Omega^2\lambda_{22}L^{1/2})|$ against $\Omega$ ($0 < \Omega \leqslant 3$) for $s/L = 0.500$, 1.000, 2.000 and 4.000 in Fig. 12. From the plots, it seems

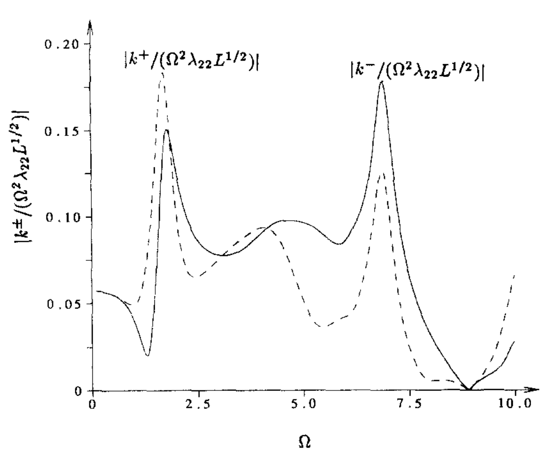


Fig. 10. Plots of $|k^{\pm}/(\Omega^2\lambda_{22}L^{1/2})|$ against $\Omega$.

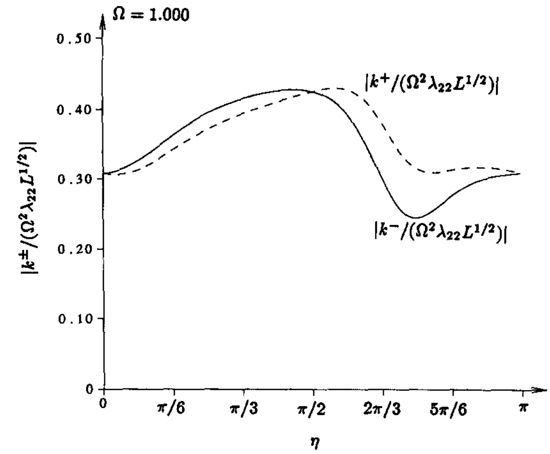


Fig. 11. Effect of varying the angle $\eta$ on $|k^\pm/(\Omega^2\lambda_{22}L^{1/2})|$.

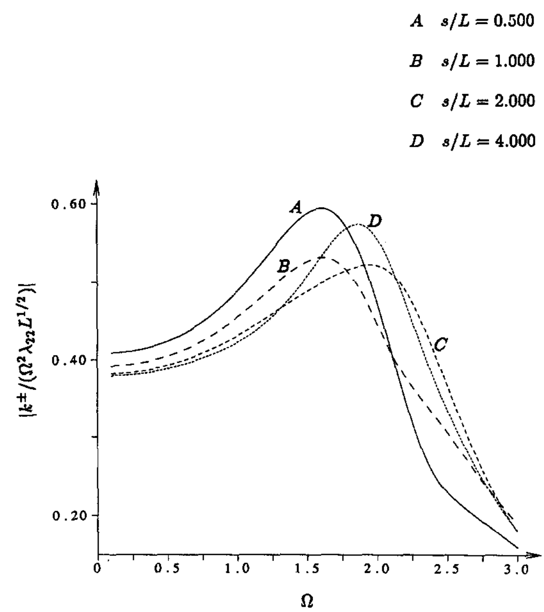


Fig. 12. Plots of $|k^\pm/(\Omega^2\lambda_{22}L^{1/2})|$ against $\Omega$ for various values of $s/L$.

that, for a given low frequency $\Omega$, $|k^+/(\Omega^2\lambda_{22}L^{1/2})|$ decreases as the distance $s/L$ (between the cracks) increases.

## 6. SOME REMARKS

We have described a numerical procedure for solving the system of hypersingular integral equations for only the case where the cracks do not intersect with one another and the plane boundary $x_2 = 0$. For edge cracks, i.e. those which intersect with the boundary $x_2 = 0$, the integral equations are still valid and can be solved by modifying (26) in an appropriate manner [see Kaya and Erdogan (1987)]. It is clear that the derivation of the hypersingular integral equations is not restricted only to planar cracks and can be easily generalized to include other arbitrarily-shaped cracks.

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